AD 741414

UNITED STATES AIR FORCE

AIR UNIVERSITY

AIR FORCE INSTITUTE OF TECHNOLOGY

Wright-Patterson Air Force Base, Ohio

Security Classification

DOCUMENT CONTROL DATA - R & D

(Security classification of title, body of abstract and indexing annotation must be entered when the overall report is classified)

1. ORIGINATING ACTIVITY (Corporate author) Air Force Institute of Technology (AFIT/EN) Wright-Patterson AFB, Ohio 45433	2a. REPORT SECURITY CLASSIFICATION Unclassified 2b. GROUP

3. REPORT TITLE

Distribution Free Methods for the Chance-Constrained Programming Model

4. DESCRIPTIVE NOTES (Type of report and inclusive dates)

AFIT Thesis

5. AUTHOR(S) (First name, middle initial, last name)

Jon R. Thomas
Captain USAF

6. REPORT DATE March 1972	7a. TOTAL NO. OF PAGES 53	7b. NO. OF REFS 31

8a. CONTRACT OR GRANT NO. b. PROJECT NO. N/A c. d.	9a. ORIGINATOR'S REPORT NUMBER(S) GSA/SM/72-15 9b. OTHER REPORT NO(S) (Any other numbers that may be assigned this report)

10. DISTRIBUTION STATEMENT

Approved for public release, distribution unlimited.

11. SUPPLEMENTARY NOTES Approved for public release; IAW AFR 190-17 Keith A. Williams KEITH A. WILLIAMS, 1st Lt, USAF Assistant Director of Information, AFIT	12. SPONSORING MILITARY ACTIVITY

13. ABSTRACT

This paper is concerned with the development of certainty or deterministic equivalent nonlinear programming models from chance-constrained programming models. It contains a review of some of the historical developments in this area which were made by Charnes and Cooper, Kataoka, Miller and Wagner, Hillier, and Sengupta. The paper introduces a new, distribution free approach to chance-constrained programming which can be used with both single and joint chance constraints. Finally, the distribution free chance-constrained model is applied to the economic problem of input-output analysis.

DD FORM 1 NOV 65 1473

UNCLASSIFIED
Security Classification

14 KEY WORDS	LINK A		LINK B		LINK C	
	ROLE	WT	ROLE	WT	ROLE	WT
Chance-Constrained Programming						
Input-Output Analysis						

DISTRIBUTION FREE METHODS

FOR THE

CHANCE-CONSTRAINED PROGRAMMING MODEL

THESIS

GSA/SM/72-15

Jon R. Thomas
Captain USAF

DISTRIBUTION FREE METHODS

FOR THE

CHANCE-CONSTRAINED PROGRAMMING MODEL

THESIS

Presented to the Faculty of the School of Engineering of

the Air Force Institute of Technology

Air University

in Partial Fulfillment of the

Requirements for the Degree of

Master of Science

by

Jon R. Thomas, B. S.

Captain USAF

Graduate Systems Analysis

March 1972

Preface

Chance-constrained programming can best be described as an attempt to optimally allocate resources in situations where the decision maker is faced with risk and uncertainty. It can be a valuable tool for the decision maker since it gives him direct control over some of this risk. In writing this thesis, I have attempted to give the analyst a review of the fundamental theory of chance-constrained programming. The thesis, however, should not be considered a literature search or a review of all the developments and applications of this theory, but it should give the analyst a thorough introduction to the subject, one which can be used to solve most problems that an analyst might encounter in this area.

However, this thesis does introduce a new distribution free approach to chance-constrained programming which was developed by R. A. Agnew. I would like to express my personal appreciation to Dr. Agnew for sharing these developments with me and also for the aid he has given me in the preparation of this paper. However, I want to point out that any errors which might appear in this thesis are solely the responsibility of the author.

Finally, I would like to express my sincerest appreciation to my wife, Dianne, for enduring the neglect that resulted from this effort.

Table of Contents

Abstract

This thesis is concerned with the development of certainty or deterministic equivalent nonlinear programming models from chance-constrained programming models. It contains a review of some of the historical developments in this area which were made by Charnes and Cooper, Kataoka, Miller and Wagner, Hillier, and Sengupta. The thesis introduces a new, distribution free approach to chance-constrained programming which can be used with both single and joint chance constraints. Finally, the distribution free chance-constrained model is applied to the economic problem of input-output analysis.

Chapter I

Introduction

Background Information

Throughout history, mankind has been concerned with the problem of optimal allocation of limited resources among competing activities. The problem has been widespread. Generals have asked the question of whether their chances of victory would be greater if they initially committed a division to combat or if they held that division in reserve to be utilized at a later stage in the battle. Farmers have wondered whether their return would be greater if they planted only corn, only wheat, or some mix of the two crops. Likewise, entrepreneurs have continually searched for more efficient ways of production in order that they might increase profits. To aid man in his search for answers to these questions, the sciences of economics, operations research, and systems analysis were developed; and it is from these sciences that man has gotten the very valuable tool of linear programming.

Linear programming was initially developed by George B. Dantzig in 1947 as a technique for planning the diversified activities of the U. S. Air Force. In any operating period the Air Force has many activities such as logistics, training, maintenance, and operations which have to be coordinated in order to achieve specific goals. Thus linear programming addresses the problem of setting optimal levels for these activities. Mathematically, the linear programming model is normally formulated in the following manner:

$$\text{maximize} \quad \Sigma_{j=1}^{n} c_j x_j$$

$$\text{subject to} \quad \Sigma_{j=1}^{n} a_{ij} x_j \leq b_i; \quad i = 1, 2, \ldots, m$$

$$x_j \geq 0; \quad j = 1, 2, \ldots, n,$$

where x_j is the j-th decision variable, i.e. the level that the decision maker chooses to set for the j-th input, product, or the j-th component of whatever the decision maker controls, and the c_j, a_{ij} and b_i are known constants. Such a model, as formulated, might carry the economic interpretation that a firm which utilizes m separate production processes in order to produce n different products is attempting to maximize total receipts from the sale of these products subject to the constraint that each production process is not operating above capacity. Here the decision maker is attempting to set optimal levels of product output. Dantzig developed the simplex algorithm to solve this linear programming problem; and since that time the model has satisfactorily been applied to not only production problems, but to problems in the areas of transportation, agriculture, and dietetics, just to name a few.

Although the linear programming model has been conceptually useful for planning in many areas, it has a major drawback. The coefficients c_j, a_{ij} and b_i are assumed to be known constants, whereas, in reality, they may neither be known nor constants. Since the model is utilized to select some future course of action, the estimates on these coefficients would be based on some prediction of future conditions. Information used in this estimation may not be sufficient to accurately predict the future values of the coefficients. Furthermore, the values of the coefficients could be greatly influenced by random events which are impossible to predict. This leads to the problem where some or all of the coefficients are actually random variables. One approach of handling this problem would be to obtain estimates of the means of the random coefficients and to use these estimates in the linear programming model. Certainly such an approach will give the decision maker some insight into his problem, but it completely ignores the random nature of the problem and suppresses the uncertainty that the decision maker faces. However, under this approach a good analyst would perform sensitivity analysis on the model; and this would give the decision maker some insight into the uncertainty he faces. But it can be safely assumed that as the number of random elements increases, the sensitivity of the solution to the model also increases, thus decreasing the value of the solution to the decision maker. However, there are other approaches that analysts can take when attempting to apply the linear programming model to situations possessing a high level of uncertainty.

Several approaches to the problem of mathematical programming under uncertainty have been developed. One approach, developed by G. Tintner, is called "stochastic linear programming" and deals with the probability distribution of $\max \Sigma_{j=1}^{n} c_j x_j$. A second approach, developed by G. Dantzig, is called "linear programming under uncertainty". This approach has successfully been applied to multi-period problems where a decision is made each period after the values of some of the random elements become known. A third approach was developed by Charnes and Cooper and is called Chance-Constrained Programming. It has the mathematical formulation

$$\max \; f(x_1, \ldots, x_n)$$

$$\text{subject to} \quad P[\Sigma_{j=1}^{n} a_{ij} x_j \le b_i] \ge 1 - \alpha_i; \; i = 1, 2, \ldots m$$

$$x_j \ge 0; \; j = 1, 2, \ldots n,$$

where again x_j is the j-th decision variable, and the a_{ij} and b_i may be random variables. In this case the objective function, $f(x_1, \ldots, x_n)$, is an attempt to quantify the decision maker's goal in terms of the decision variables. Also, $P[\Sigma_{j=1}^{n} a_{ij} x_j \le b_i] \ge 1 - \alpha_i$ is interpreted to mean that the probability that the i-th constraint is violated is less than α_i, an appropriately chosen number between zero and one. If $f(x_1, \ldots, x_n)$ is a linear function, one can see that the formulation of the chance-constrained model closely resembles the linear programming model, but it changes the i-th constraint from a linear inequality into a probability statement, thus recognizing the uncertainty associated with the problem.

There is an important distinction between the chance-constrained model and most other constrained optimization problems since the violation of the constraints is permissible in the chance-constrained approach. One may question the merit of allowing a constraint to be violated, but it can be demonstrated that this is a valuable practice when dealing with problems which contain elements subject to random variation. To illustrate this point, suppose one is producing a product with the constraint that the random demand for that product must always be met, i.e., demand is met with probability one. But such a policy says that the probability that production exceeds demand is one or very close to one, and if the variance of this random demand is large, over production may also get large causing storage costs to rise, thus cutting into profits. The chance-constrained programming approach to this problem would permit the decision maker to specify the level of risk of not meeting demand that he is willing to face. This is done by choosing an appropriate α_i. It is in this light that one sees the value of using the chance-constrained programming model, since it forces the decision maker to recognize the risk he faces and allows him to choose the level of risk he feels appropriate for the problem.

Consider again the i-th chance-constraint which is written as

$$P[\Sigma_{j=1}^{n} a_{ij}x_j \leq b_i] \geq 1 - \alpha_i.$$

Since the a_{ij} and b_i may be random variables (it is assumed that there is at least one random variable in each chance-constraint), it is clear that the chance-constraint cannot be directly utilized in solving the chance-constrained programming model. It is then necessary that one transform the chance-constraint into a deterministic equivalent constraint which is free of any random variables before the model can be solved. Most of the literature written in this area has addressed this problem, but in many cases, the authors made assumptions about the probability distributions of the random variables a_{ij} and b_i. These assumptions have limited the applicability of their deterministic equivalent model.

The Problem

J. K. Sengupta and S. M. Sinha have utilized the Chebyshev inequality in chance-constrained programming. The inequality has a particularly useful characteristic in that it holds regardless of the probability distribution of the random variable being measured. The purpose of this paper is to utilize the Chebyshev inequality to develop deterministic equivalent constraints for single chance-constraints and joint chance-constraints, a concept developed by Miller and Wagner. The model developed will be general in that it will allow any of the coefficients to be random variables with unknown probability distributions. The model will then be applied to the economic problem of input-output analysis to demonstrate its usefulness to the decision maker.

Overview

Chapter II of this paper will illustrate some of the approaches that other authors have taken in solving the chance-constrained programming model. The assumption will initially be made that the probability distributions of the random variables involved are known. Then deterministic equivalent constraints will be developed. This assumption will then be relaxed. Also, Miller and Wagner's concept of a joint chance-constraint will be reviewed. Finally, J. K. Sengupta's approach to chance-constrained programming which utilizes the Chebyshev inequality will be reviewed.

Chapter III will extend the use of the Chebyshev inequality to give deterministic equivalent constraints which can be used to solve the most general case of the chance-constrained programming model which has both single and joint chance-constraints.

Chapter IV will then apply the results of chapter III to the economic problem of input-output analysis. The input-output analysis model will also be modified to show how it might be of use to the Air Force and other Department of Defense planners.

Chapter II

Historical Developments

The chance-constrained programming model is normally formulated as

$$\max \quad f(x_1, \ldots, x_n)$$

$$\text{subject to} \quad P[\Sigma_{j=1}^{n} a_{ij}x_j \geq b_i] \geq 1 - \alpha_i \quad i = 1, 2, \ldots, m$$

$$x_j \geq 0 \quad j = 1, 2, \ldots, n, \qquad (1)$$

where the x_j are the non-stochastic decision variables, the a_{ij} and b_i may be random variables, and the α_i is an appropriately chosen number which lies between 0 and 1. The i-th probabilistic constraint allows that constraint to be violated with probability α_i; hence α_i represents the allowable risk which the decision maker chooses to face. This formulation additionally requires that the n decision variables all be non-negative, which is generally the case for most economic problems. However, one could easily restrict the feasible region of choice for the decision variables by including other non-probabilistic constraints in the form of linear inequalities, where constants and the coefficients of the decision variables are non-stochastic.

As formulated, the chance-constrained programming model cannot be directly solved, but each of the probabilistic constraints must be reduced to certainty equivalent or deterministic equivalent constraints. This is accomplished by using the means and variances of the random variables given in the chance-constraint and developing certainty equivalent constraints which are usually in the form of non-linear inequalities. Before investigating some of the approaches taken by various authors in this area, a few words should be said about an appropriate choice for the objective function, $f(x_1, \ldots, x_n)$.

The Objective Function

In choosing an objective function to be used in the chance-constrained programming model, an analyst should attempt to quantify the decision maker's major objective. This usually gives some mathematical function in terms of the decision variables, $x_1, \ldots, x_n$. It is seen in the linear programming model that the objective function is given by

$$\max \ \Sigma_{j=1}^{n} c_j x_j,$$

and this objective function could carry the interpretation that the firm is attempting to maximize receipts from the sale of goods produced. Here c_j represents the market price for product j. This certainly appears to be a logical choice for a firm's major objective. Clearly, this objective function can be utilized in the chance-constrained programming model. However, since the firm is operating in an environment of uncertainty, it appears quite likely that some of this uncertainty will be encountered in the market; hence each price, c_j, could very possibly be a random variable. It would be ambiguous to deal with max $\Sigma_{j=1}^{n} c_j x_j$ when the c_j are random variables. As is the case with the probabilistic constraint, one must develop a certainty equivalent expression for receipts in order to use maximum receipts as the objective function. Since receipts are normally a linear function of the decision variables, and programming algorithms using a linear objective function deal primarily with a relative weighting of the decision variables, it seems logical to utilize the expected value of receipts as the appropriate objective function. Since the decision variables are assumed to be non-stochastic, this is given by

$$E(\Sigma_{j=1}^{n} c_j x_j) = \Sigma_{j=1}^{n} E(c_j) x_j,$$

where $E(c_j)$ is the expected market price for product j. The chance-constrained programming model which uses this objective function is referred to as the "E Model", a term given by Charnes and Cooper, the developers of the model.

Use of the E Model will be satisfactory if the decision maker or the firm has a linear utility for wealth, but, in reality, most individuals have a diminishing marginal utility. The logarithmic utility function is a good example of this. Therefore, if the decision maker does not have a linear utility for wealth, the analyst must use some function which represents the expected utility of profits as an objective function. Since profit is a random variable whose distribution may be known or unknown, it might be difficult to obtain an expression for the expected utility of profits.

However, due to the arbitrary nature of most utility functions, it would be permissible to use a tractible lower bound as the objective function in place of the expected utility function. R. A. Agnew, [2] and [3], has developed such bounds, and they depend only on the mean and variance of random profits. The chance-constrained programming model which uses a bound for the expected utility of wealth can readily be applied to the problem of portfolio selection [1], [25], and [26].

Suppose the decision maker is more concerned with minimizing the random fluctuations of receipts, as opposed to maximizing expected receipts. In this case, the analyst would use min Var $(\Sigma_{j=1}^{n} c_j x_j)$ as the objective function. Fortunately, this variance function is a positive semi-definite quadratic form with respect-to the decision variables, which means a global minimum can be found for this function if the decision variables are confined to a convex set. The chance-constrained programming model which uses this objective function is normally called the "V Model", a term originated by the developers of the model, Charnes and Cooper. Realistically, one would not usually see the V Model used without an additional constraint that expected receipts are greater than some specified level that the decision maker feels is satisfactory.

Charnes and Cooper developed another chance-constrained programming model which they called the "P Model". In this case the decision maker determines some satisfactory level of receipts, call it $\bar{Z}$, and then attempts to maximize the probability that receipts are equal to or greater than this level. The authors call the utilization of this model satisficing, a term developed by H. A. Simon. Mathematically this can be expressed as

$$\max \quad P[\Sigma_{j=1}^{n} c_j x_j \geq \bar{Z}].$$

This model may take on a different interpretation if one considers $\bar{Z}$ to be the minimum acceptable level of receipts or, as some authors call it, the "disaster level". In this case, the objective is to minimize the probability of ruin or

$$\min \quad P[\Sigma_{j=1}^{n} c_j x_j < \bar{Z}].$$

Neither of these expressions can be used as a deterministic objective function to solve the model. If one defines the functions

$$\mu_o = \Sigma_{j=1}^{n} E(c_j)x_j \text{ and } \sigma_o^2 = \text{Var}\,(\Sigma_{j=1}^{n} c_j x_j),$$

it can be shown that it is mathematically equivalent in both these cases to maximize the following certainty equivalent objective function

$$(\mu_o - \bar{Z})/\sigma_o,$$

provided that the cumulative distribution function of $(\Sigma_{j=1}^{n} c_j x_j - \mu_o)/\sigma_o$ is strictly monotonic. This objective function would be utilized if the decision maker's major objective is to minimize the risk he faces and is usually referred to as the safety-first approach.

Along these same lines, Kataoka [22] proposed a slight modification to the above model which has also become known as the safety-first approach. This model is formulated as

$$\max \bar{Z}$$

$$\text{subject to} \quad P[\Sigma_{j=1}^{n} c_j x_j < \bar{Z}] \leq \alpha.$$

This can be shown to be mathematically equivalent to maximizing

$$\mu_o + F^{-1}(\alpha)\sigma_o,$$

where F is the cumulative distribution function of the random variable $(\Sigma_{j=1}^{n} c_j x_j - \mu_o)/\sigma_o$.

It can be clearly seen that the analyst has a great deal of flexibility in modeling the decision maker's attitude by using one of the above objective functions and using others to make realistic constraints.

Development of Deterministic Constraints

Having chosen an appropriate objective function, the analyst must now transform the probabilistic constraints given in the chance-constrained programming model into deterministic or certainty equivalent constraints. Consider the i-th constraint from (1), which is given by

$$P[\Sigma_{j=1}^{n} a_{ij} x_j \leq b_i] \geq 1 - \alpha_i.$$

Let $y_i = \Sigma_{j=1}^{n} a_{ij} x_j - b_i$, $\mu_i = E(y_i)$, and $\sigma_i^2 = \text{Var}(y_i)$. Remember that u_i and σ_i^2 are nonstochastic functions of the decision variables. Assume that there is at least one random variable in each constraint, which means that $\sigma_i^2 > 0$, and that the random variable y_i has a known probability distribution with finite mean and variance. Using these assumptions the model can be transformed into the following deterministic equivalent model*:

---

*See Appendix A for the mathematical development.

$$\begin{aligned}
\max \quad & f(x_1, \ldots, x_n) \\
\text{subject to} \quad & \mu_i + K_{1-\alpha_i} v_i \leq 0 \\
& v_i^2 - \sigma_i^2 \geq 0 \quad i = 1, 2, \ldots, m \\
& v_i \geq 0 \\
& x_j \geq 0 \quad j = 1, 2, \ldots, n, \qquad (2)
\end{aligned}$$

where μ_i and σ_i^2 are defined as above and

$$K_{1-\alpha_i} = F_i^{-1}(1 - \alpha_i);$$

F_i represents the cumulative distribution function of the random variable $(y_i - \mu_i)/\sigma_i$. When one utilizes either the E Model or V Model objective function, (2) yields a convex programming problem, and there are algorithms available to solve this problem. If one utilizes the P Model objective function, the problem is not quite as tractible. Remember that the certainty equivalent objective function for the P Model is given by max $(\mu_o - \bar{Z})/\sigma_o$ where $\mu_o = E(\Sigma_{j=1}^{n} c_j x_j)$ and $\sigma_o^2 = \text{Var}(\Sigma_{i=1}^{n} c_j x_j)$, both functions of the decision variables. Utilizing this, the deterministic equivalent for the P Model chance-constrained programming model becomes:

$$\begin{aligned}
\max \quad & v_o/w_o \\
\text{subject to} \quad & \mu_o - \bar{Z} - v_o \geq 0 \\
& w_o^2 - \sigma_o^2 \geq 0 \\
& \mu_i + K_{1-\alpha_i} v_i \leq 0 \\
& v_i^2 - \sigma_i^2 \geq 0 \quad i = 1, 2, \ldots, m \\
& v_i \geq 0 \\
& v_o, w_o \geq 0 \\
& x_j \geq 0 \quad j = 1, 2, \ldots, n. \qquad (3)
\end{aligned}$$

In this case the objective function v_o/w_o is neither concave nor convex. But the problem can be solved using fractional programming techniques, and as a consequence of the theory in this area, a local maxima will in fact be a global maxima for linear fractional functionals.

If the random variables, $a_{i1}, \ldots, a_{in}, b_i$, have joint normal distribution, the value for $K_{1-\alpha_i}$ can be found in the cummulative tables of the standard normal variate. However, if the random variables in each constraint have arbitrary distributions, y_i may still be approximated by the normal distribution since some version of the Central Limit Theorem may hold. This theorem holds under weak conditions for independent random variables and under stronger conditions for dependent random variables*. Suppose the distribution of y_i is not known and the Central Limit Theorem cannot be applied to y_i. Hillier [18] suggests using the one-sided Chebyshev inequality which gives $[(1-\alpha_i)/\alpha_i]^{\frac{1}{2}}$ as an upper bound on $K_{1-\alpha_i}$, and this value can be used in the deterministic models (2) and (3).

Joint Constraints

The concept of a joint chance-constraint was first introduced by Miller and Wagner [24]. In this case the model is formulated as

---

*For a survey of conditions under which the Central Limit Theorem holds, see Hillier, [20].

$$\max \quad f(x_1, \ldots, x_n)$$

$$\text{subject to} \quad P \cap_{i=1}^{m} [\Sigma_{j=1}^{n} a_{ij}x_j \le b_i] \ge 1 - \alpha$$

$$x_j \ge 0 \quad j = 1, 2, \ldots, n.$$

In this formulation, all of the chance-constraints from (1) are joined together into one joint constraint. This formulation carries the interpretation that the probability that any of the m constraints is violated is less than α, an appropriately chosen number between zero and one. In their development, Miller and Wagner first assume that all the a_{ij} are non-stochastic and that b_i and b_k, $i \ne k$, are stochastically independent random variables with known, continuous distributions. This implies that the joint chance-constraint can be represented as

$$P \cap_{i=1}^{m} [\Sigma_{j=1}^{n} a_{ij}x_j \le b_i] = \pi_{i=1}^{m} P[\Sigma_{j=1}^{n} a_{ij}x_j \le b_i].$$

Letting $F_i(\cdot)$ represent the cumulative distribution function of the random variable b_i, they define

$$G_i(b_i) = 1 - F_i(b_i).$$

Their model can then be expressed as the following nonlinear programming problem:

$$\max \quad f(x_1, \ldots, x_n)$$

$$\text{subject to} \quad -v_i + \Sigma_{j=1}^{n} a_{ij}x_j \le 0 \quad i = 1, 2, \ldots, m$$

$$\pi_{i=1}^{m} G_i(v_i) \ge 1 - \alpha$$

$$x_j \ge 0 \quad j = 1, 2, \ldots, n.$$

Notice in this case that v_i is unconstrained in sign, and the authors remark that for computational purposes either $-v_i + \Sigma_{j=1}^{n} a_{ij}x_j = 0$ or $\pi_{i=1}^{m} G_i(v_i) = 1 - \alpha$, but generally not both. If $\pi_{i=1}^{m} G_i(v_i)$ is a concave function, then a global optimum can be found by using some nonlinear programming algorithm. The major portion of Miller and Wagner's paper addresses the question of when $\pi_{i=1}^{m} G_i(v_i)$ is concave. They could not develop conditions which are sufficient for $\pi_{i=1}^{m} G_i(v_i)$ to be concave; however, they were able to determine that it will not be concave when all the b_i have normal, gamma, and uniform distributions. But if one makes the following transformation

$$\Sigma_{i=1}^{m} \ln G_i(v_i) \geq \ln (1 - \alpha),$$

the constraint defines a convex region when the distribution of each b_i is uniform or normal. This transformation also works for the gamma and Weibull densities, when their parameter θ is greater than one. In this case the model is formulated as

$$\begin{aligned} \max \quad & f(x_1, \ldots, x_n) \\ \text{subject to} \quad & -v_i + \Sigma_{j=1}^{n} a_{ij}x_j \leq 0 \quad i = 1, 2, \ldots, m \\ & \Sigma_{i=1}^{m} \ln G_i(v_i) \geq \ln (1 - \alpha) \\ & x_j \geq 0 \quad j = 1, 2, \ldots, n. \end{aligned}$$

The above arguments can be directly applied to the case where one has several joint constraints if each constraint appears in only one joint constraint; that is, there is no constraint, $\Sigma_{j=1}^{n} a_{ij}x_j \leq b_i$, which appears in more than one joint constraint.

Miller and Wagner also consider the case where all the a_{ij} have normal distribution and the b_i are non-stochastic. Again the assumption that all random variables are stochastically independent is made. Let $y_i = \Sigma_{j=1}^{m} a_{ij}x_j$ where

$\mu_i = E(y_i)$ and $\sigma_i^2 = \text{Var}\ (y_i)$. In this case, the joint constraint becomes

$$\mu_i + v_i\sigma_i \geq b_i \quad i = 1, 2, \ldots, m$$

$$\Sigma_{i=1}^{m} \ln G(v_i) \geq \ln (1 - \alpha)$$

where $G(v_i) = 1 - F(v_i)$; $F(.)$ is the cumulative distribution function of the standard normal variate. Clearly, these constraints do not define a convex region, so any algorithm which solves this problem may give only a local optimum and not a global one.

Safety-First Approach to Chance-Constrained Programming

An approach to chance-constrained programming is available in the safety-first principle which is best described by A. D. Roy, [28], when he said, "...it is reasonable, and probable in practice, that an individual will seek to reduce as far as is possible the chance of such a catastrophe occurring." Roy defines a catastrophe as an event where an individual makes a net loss on some investment or venture. Such an approach can be expressed as the "P Model" of the chance-constrained programming problem which is formulated as

$$\min \quad P[\Sigma_{j=1}^{n} c_j x_j \leq \bar{Z}]$$

$$\text{subject to} \quad P[\Sigma_{j=1}^{n} a_{ij} x_j \leq b_i] \geq 1 - \alpha_i \quad i = 1, 2, \ldots, m$$

$$x_j \geq 0 \quad j = 1, 2, \ldots, n,$$

where $\bar{Z}$ is the "disaster level" of receipts.

J. K. Sengupta [29] has applied the distribution free Chebyshev inequality to this problem by using an approach somewhat different than those previously reviewed in this chapter. In his approach, Sengupta uses an idea attributed to Theil where there is a penalty cost associated with each constraint. This penalty cost is only applied when the constraint is violated. Also, there is a penalty cost which is applied whenever receipts are less than or equal to the disaster level, $\bar{Z}$. Sengupta then defines a new objective function which can be called total expected penalty costs, thus transforming the constrained problem into an unconstrained minimization problem. In this approach, the decision maker must be able to assign a realistic penalty cost to each constraint.

To quantify this, let h be the unit penalty cost to be applied whenever $\bar{Z}$ exceeds $\Sigma_{j=1}^{n} c_j x_j$, and k_i be the unit penalty cost to be applied whenever $\Sigma_{j=1}^{n} a_{ij} x_j$ exceeds b_i. Sengupta assumes that the a_{ij} are non-stochastic constants and that the c_j and b_i are continuous random variables defined on a non-negative range. He defines $C(x_1, \ldots, x_n)$ to be the expected penalty cost associated with the original objective function and $C_i(x_1, \ldots, x_n)$ to be the expected penalty cost associated with the i-th constraint. The problem can now be expressed as

$$\min \quad C(x_1, \ldots, x_n) + \Sigma_{i=1}^{m} C_i(x_1, \ldots, x_n)$$

$$\text{subject to} \quad x_j \geq 0 \quad j = 1, 2, \ldots, n.$$

The expected penalty cost associated with the i-th constraint would be given by

$$C_i(x_1, \ldots, x_n) = k_i \int_0^{s_i} (s_i - b_i)\, f_i(b_i)\, db_i,$$

where $s_i = \Sigma_{j=1}^{n} a_{ij}x_j$ and $f_i(b_i)$ is the probability density function of the random variable b_i. If b_i has unknown distribution, this integral would be impossible to evaluate. However, Sengupta uses the Chebyshev inequality to obtain an upper bound for the integral, and the bound can be expressed as

$$C_i(x_1, \ldots, x_n) \leq k_i(\Sigma_{j=1}^{n} a_{ij}x_j)\sigma_i^2/\mu_i^2,$$

where $\sigma_i^2 = \text{Var}(b_i)$ and $\mu_i = E(b_i - \Sigma_{j=1}^{n} a_{ij}x_j)$. Similarly one can derive an upper bound on the expected penalty cost associated with the objective function,

$$C(x_1, \ldots, x_n) \leq h\,\bar{Z}\,\sigma_o^2/(\mu_o - \bar{Z})^2,$$

where $\sigma_o^2 = \text{Var}(\Sigma_{j=1}^{n} c_jx_j)$ and $\mu_o = E(\Sigma_{j=1}^{n} c_jx_j)$.

Following the safety-first approach to chance-constrained programming, Sengupta would then solve the following problem

$$\min \; h\,\bar{Z}\,\sigma_o^2/(\mu_o - \bar{Z})^2 + \Sigma_{i=1}^{m} k_i(\Sigma_{j=1}^{n} a_{ij}x_j)\sigma_i^2/\mu_i^2$$

subject to $x_j \geq 0 \quad j = 1, 2, \ldots, n.$

Chapter III

A Distribution Free Approach to Chance-Constrained Programming

In the last chapter, some of the developments in the theory of solution techniques for the chance-constrained programming model were reviewed. This included the introduction of the concept of a joint chance-constraint. Also, it was demonstrated how the distribution free Chebyshev inequality can be utilized to give deterministic bounds on probabilities which can be used in the safety-first approach to chance-constrained programming.

To continue in this fashion, this chapter will be concerned with the utilization of the one-sided Chebyshev inequality to give deterministic constraints which can be used to solve the chance-constrained programming model which has both single and joint chance-constraints. Miller and Wagner's development of the joint constraint requires independence of the random variables involved, whereas the distribution free techniques developed in this chapter will relax this restriction. The concepts outlined in this chapter are developed from ideas given to the author by R. A. Agnew; therefore, all credit for the development should be given to Dr. Agnew.

Before proceeding to the development of the distribution free techniques, one should consider the form of the one-sided Chebyshev inequality to be used in this development. It is taken from Feller [16]. Given a random variable x with $E(x) = 0$ and $Var(x) = \sigma^2$, and given any $t \geq 0$, then $P[x > t] \leq \sigma^2/(\sigma^2 + t^2)$.

Proceeding to the development of a distribution free deterministic equivalent program for the chance-constrained programming model, let $(\Omega, \mathcal{Q}, P)$ be a probability space and let $[y_i = y_i(x_1, \ldots, x_n, \omega),\ i = 1, 2, \ldots, m]$ be a family of random variables from this space which are defined in terms of the non-stochastic decision variables. Assume that $E(y_i) < \infty$ for feasible x_j, that $E(y_i) = \mu_i$ is concave and $Var(y_i) = \sigma_i^2$ is convex for feasible x_j. One must remember that μ_i and σ_i^2 are functions of the decision variables, $x_1, \ldots, x_n$. Clearly, these assumptions hold when one deals with linear chance-constraints. With this notation in mind, the chance-constrained programming model can be expressed as

$$\begin{aligned} &\max \quad f(x_1, \ldots, x_n) \\ &\text{subject to} \quad P[y_i \geq 0] \geq 1 - \alpha_i \quad i = 1, 2, \ldots, m \\ &P\cap_{i=1}^{m} [y_i \geq 0] \geq 1 - \alpha \\ &x_j \geq 0 \quad j = 1, 2, \ldots, n, \end{aligned} \tag{4}$$

where f is some concave function and $\alpha, \alpha_1, \ldots, \alpha_m$ are suitable elements of the interval (0, 1).

In this particular formulation, there are m single chance-constraints which are also combined into one joint constraint. The model is formulated in this fashion for ease of development and presentation, but it can be generalized to handle any number of single and joint constraints by directly applying the concepts that will be developed.

Using the one-sided Chebyshev inequality and the subadditivity property of the probability measure, one can transform the chance-constrained model, (4), into the following deterministic equivalent model:

$$
\begin{aligned}
&\max \quad f(x_1, \ldots, x_n) \\
&\text{subject to} \quad t_i \le \alpha_i \\
&\qquad\qquad\quad \mu_i \ge 0 \qquad i = 1, 2, \ldots, m \\
&\Sigma_{i=1}^{m} t_i \le \alpha \\
&\qquad\qquad\quad x_j \ge 0 \quad j = 1, 2, \ldots, n, \qquad (5)
\end{aligned}
$$

where $t_i = \sigma_i^2/(\sigma_i^2 + \mu_i^2)$.**

The problem as formulated may have no feasible solution for arbitrarily stipulated $\alpha, \alpha_1, \ldots, \alpha_m$. However, one can obtain a suitable set of α's by first solving

$$
\begin{aligned}
&\min \quad \Sigma_{i=1}^{m} u_i \\
&\text{subject to} \quad t_i \le u_i \qquad i = 1, 2, \ldots, m \\
&\mu_i \ge 0, \; u_i \ge 0 \\
&\qquad\qquad\quad x_j \ge 0 \quad j = 1, 2, \ldots, n, \qquad (6)
\end{aligned}
$$

and then putting $\Sigma_{i=1}^{m} u_i^* \le \alpha$, $u_i^* \le \alpha_i$ where $u_i^*, \ldots, u_m^*$ is an optimal solution of (6).

If one combines (5) and (6) into one problem and adds variables to make the model computationally more tractible, the deterministic equivalent chance-constrained programming model becomes

$$\max \quad f(x_1, \ldots, x_n)$$

**Mathematical development is contained in Appendix A.

$$\begin{aligned} \text{subject to}\quad \mu_i &\geq w_i v_i \\ v_i^2 &\geq \sigma_i^2 \\ u_i(w_i^2 + 1) &\geq 1 \qquad\qquad i = 1, 2, \ldots, m \\ u_i &\leq \alpha_i \\ \Sigma_{i=1}^{m} u_i &\leq \alpha \\ u_i, v_i, w_i &\geq 0 \\ x_j &\geq 0 \quad j = 1, 2, \ldots, n. \end{aligned} \tag{7}$$

Unfortunately, the above constraints do not define a convex region, so the problem is not a convex programming problem. This means that any nonlinear programming algorithm used to solve this problem may converge to only a local maximum and not to a global one. But fortunately, the constraints and the objective function are differentiable, which means that there is a nonlinear programming algorithm which can be used to find a local maximum [17]. Furthermore, it will be assumed that the decision maker, who is trying to optimally achieve his objective in an environment of uncertainty and risk, will be satisfied with a locally optimal choice for the decision variables x_j; at least, until a better choice can be found.

As mentioned previously, this model can be generalized to handle any number of single and joint constraints. Suppose one has the chance-constrained programming model with only single chance-constraints. The analyst would utilize the deterministic model, (7), except he would delete the constraint

$$\Sigma_{i=1}^{m} u_i \leq 1 - \alpha.$$

If the model were given with only the joint chance-constraint, the analyst would delete the constraints

$$u_i \le \alpha_i \text{ for } i = 1, 2, \ldots, m.$$

Specifically, (7) can be applied to the linear chance-constrained programming model with both single and joint constraints. In this case, the model would be formulated as

$$\max \quad f(x_1, \ldots, x_n)$$

$$\text{subject to} \quad P[\Sigma_{j=1}^n a_{ij}x_j \le b_i] \ge 1 - \alpha_i \quad i = 1, 2, \ldots, m$$

$$P\cap_{i=1}^m [\Sigma_{j=1}^n a_{ij}x_j \le b_i] \ge 1 - \alpha$$

$$x_j \ge 0 \quad j = 1, 2, \ldots, n.$$

Then, the y_i, μ_i, and σ_i^2 would be given by

$$y_i = b_i - \Sigma_{j=1}^n a_{ij}x_j,$$

$$\mu_i = E(b_i) - \Sigma_{j=1}^n E(a_{ij})x_j, \text{ and}$$

$$\sigma_i^2 = \Sigma_{j=1}^n \Sigma_{k=1}^n \text{cov } (a_{ij}, a_{ik})x_jx_k$$

$$-2\Sigma_{j=1}^n \text{cov } (a_{ij}, b_i)x_j + \text{var } (b_i).$$

If one has estimates of the means, variances and covariances of the random variables involved, they can be put into these equations and substituted into (7). Moreover, if estimates of the covariances are not known, one can use the following bound

$$\sigma_i \le \Sigma_{j=1}^n (\text{var } (a_{ij}))^{\frac{1}{2}} x_j + (\text{var } (b_i))^{\frac{1}{2}},$$

in the constraint $v_i^2 \ge \sigma_i^2(x)$ to get the following constraint

$$v_i \ge \Sigma_{j=1}^n (\text{var } (a_{ij}))^{\frac{1}{2}} x_j + (\text{var } (b_i))^{\frac{1}{2}}.$$

Although in some cases the Chebyshev inequality may give a rough bound on a probability measure, it has a desirable property that it holds regardless which probability distribution the random variables involved may have. The inequality is particularly useful when adapted to sums of random variables which is the case with the linear chance-constrained programming model. Furthermore, it has been demonstrated that the inequality can be used to develop a deterministic equivalent model which can solve the most general case of the chance-constrained programming model which has both single and joint chance-constraints.

Chapter IV

Chance-Constrained Input-Output Analysis

Now that a distribution free, deterministic equivalent model for the chance-constrained programming model has been developed, attention will now be turned to the area where this model will be applied--Leontief's input-output analysis. Input-output analysis is well suited to the chance-constrained programming approach since input-output analysis, in its simplest form, is a linear programming problem where the constants and coefficients of the model are random variables, usually with unknown distributions. Therefore, the chance-constrained programming approach will give the decision maker everything the normal approach to input-output analysis gives, plus further insight into the risk and uncertainty that the decision maker faces. Furthermore, it will enable him to directly control some of this risk. Before proceeding with the development of a chance-constrained, input-output model, a few words will be said about the theory of input-output analysis.

Input-Output Analysis

Consider an economy which has n sectors where each sector is producing one, and only one, distinct product. Output from any particular sector may be needed as an input for itself and for other sectors. Furthermore, the open model will be considered, which allows outside demand to occur; this could be interpreted to include consumer demand, demand generated by the government, and demand for export. Also, assume that there is a finite labor supply and that each sector demands labor in order to produce goods. In order to have a viable economy, total production of product i must be greater than or equal to the total demand for product i, which includes outside demand and the demand from each sector. Furthermore, total demand for labor must not exceed the total labor supply.

To quantify this, let x_i be the total number of units produced of product i. Let a_{ij} be the number of units of product i needed to produce one unit of product j. Also, let a_{oj} be the number of units of labor needed to produce one unit of product j; L represents the total amount of labor available in the labor market. Finally, let D_i represent the outside demand for product i. The production and labor constraints can now be represented as

$$x_i - \Sigma_{j=1}^{n} a_{ij}x_j \geq D_i \quad i = 1, 2, \ldots, n$$

$$\Sigma_{j=1}^{n} a_{oj}x_j \leq L.$$

Letting c_i be the production cost of product i, the input-output analysis model can be represented by the following linear program

$$\min \ \Sigma_{j=1}^{n} c_j x_j$$

$$\text{subject to} \quad x_i - \Sigma_{j=1}^{n} a_{ij}x_j \geq D_i \quad i = 1, 2, \ldots, n$$

$$\Sigma_{j=1}^{n} a_{oj}x_j \leq L$$

$$x_j \geq 0 \quad j = 1, 2, \ldots, n.$$

This model, as formulated, can be interpreted to mean that economic planners are attempting to minimize production costs, subject to the constraints that production in each sector exceeds total demand and that total demand for labor does not exceed the total labor available.

Leontief's model is important since it recognizes that a very considerable portion of the effort of a modern economy is devoted to the production of intermediate goods, and the output of intermediate goods is closely linked to the output of final products. Therefore, a change in the level of output of some final product can cause widespread changes throughout the economy in the level of production of intermediate goods. In other words, input-output analysis describes a national economy as a system of mutually interrelated sectors or interdependent economic activities. This interrelation actually consists of flows of goods and services which directly or indirectly link all the sectors of the economy together. Leontief's approach is particularly important since it highlights this interdependence of the economic sectors. Another important feature of Leontief's model is that it is strongly dependent on empirical investigation.

The model has been successfully applied to the United States economy, and large tables have been constructed which show how goods and services flow throughout the U. S. economy. The model can be of great use for economic planning in developing nations since it can be utilized to predict the "useful" production capability of that nation. In the same light, it can be used for planning optimal levels of economic growth. Furthermore, the model can be an aid for economic planning in military mobilization problems.

Although Leontief developed the model for a large economic system, it can be readily seen that input-output analysis can be applied to problems much smaller in scope. The model can be applied to any organization or system that is composed of components which supply goods and services to other components within the organization or system. Furthermore, these goods and services may be supplied to activities outside the organization. Clearly, this would include large manufacturing organizations and a military activity like the U. S. Air Force.

Baumol [5] points out two simplifying assumptions made in input-output analysis which have come under criticism. One is the assumption that each sector manufactures only one homogeneous product or good. This restriction may be relaxed somewhat by interpreting this product as a composite commodity. The second and more restrictive assumption made is that all inputs needed in the manufacturing of a unit of output are utilized in fixed proportions. This does not allow the possibility of trade-offs or substitution of inputs. Clearly, adoption of the chance-constrained approach will relax this second assumption.

Another drawback against using input-output analysis is that a great amount of statistical effort must be expended in order to come up with timely forecasts of demand and estimators of the technical coefficients, a_{ij}. It might be pointed out that this statistical effort could also provide an estimation of the variance of these estimators without much greater effort being expended.

The Model

The chance-constrained, input-output analysis model will now be developed, and it deviates from the usual economic interpretation since two military units will be considered as the economic sectors. These will be an airlift command which will represent sector I and a logistics command representing sector II. Now sector I, the airlift command, provides a service whose unit of measure is a ton of materiel/personnel transported. Sector I provides this service to itself, to sector II, and to other military activities. Likewise, sector II provides a product whose unit of measure is a ton of materiel supplied. Again it uses the product itself, and the product is used by sector I and by other military activities.

To quantify this problem, let x_1 represent the total amount of airlift support provided by sector I. Let a_{11} be the random variable which represents the amount of airlift support required to supply one ton of materiel/personnel transported, and a_{12} represents the random amount of airlift support needed to provide one ton of materiel supplied. Let D_1 represent the demand for airlift support by all other military activities. Similarly, let a_{21}, a_{22}, and D_2 be the random demands for logistic support, and let x_2 be the total amount of logistic support provided by sector II. To restrict the model, assume that x_1 and x_2 are non-stochastic decision variables. This is not too an unrealistic assumption if one interprets x_1 and x_2 to be the upper bound on the airlift and logistic support that decision makers are willing to provide. Furthermore, let c_1 and c_2 be the random unit cost in thousands of dollars of airlift support and logistic support. Let 100 be the number of manhours required to provide one ton of materiel/personnel transported, 50 be the number of manhours required to provide one ton of materiel supplied, and 650,000 the total number of manhours available to both the airlift command and the logistics command. Finally, assume that decision makers require that the demand for airlift support will be met with at least probability .95; also, they require that the demand for logistic support will be met with at least probability .9. Lastly, decision makers require that the total demand for both airlift support and logistic support will be met with at least probability .9.

The chance-constrained, input-output analysis model is now given by

$$\min \quad E(c_1x_1 + c_2x_2)$$

$$\text{subject to} \quad P[(1 - a_{11})x_1 - a_{12}x_2 \geq D_1] \geq .95$$

$$P[-a_{21}x_1 + (1 - a_{22})x_2 \geq D_2] \geq .9$$

$$P\left[\begin{bmatrix}(1 - a_{11}) & -a_{12}\\ -a_{21} & 1 - a_{22}\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} \geq \begin{bmatrix}D_1\\ D_2\end{bmatrix}\right] \geq .9$$

$$100x_1 + 50x_2 \leq 650{,}000$$

$$x_1,\ x_2 \geq 0,$$

where the objective is taken to be the minimization of expected costs.

Suppose statistical analysis gives the following values for estimators of the mean and variance of the above random variables.

Random Variable	Mean	Variance
c_1	35	5
c_2	25	3
a_{11}	.25	.0025
a_{12}	.3	.0009
a_{21}	.15	.0001
a_{22}	.2	.0004
D_1	1000	10,000
D_2	1500	12,000,

and the covariances are given by

$$\sigma_{a_{11},\, a_{12}} = -.0009$$

$$\sigma_{a_{21},\, a_{22}} = -.0001.$$

All other covariances are zero.

Following the non-stochastic approach to input-output analysis, Department of Defense planners would take the estimators of the means of the random variables, and solve the following linear programming model.

$$\min \quad 35x_1 + 25x_2$$

$$\begin{aligned} \text{subject to} \quad .75x_1 - .3x_2 &\geq 1000 \\ -.15x_1 + .8x_2 &\geq 1500 \\ x_1 + .5x_2 &\leq 6500 \\ x_1,\ x_2 &\geq 0 \end{aligned}$$

Solving this problem, one obtains the solution $x_1 = 2252.252$ and $x_2 = 2297.297$, which gives total expected cost of $136,261,300. Not much can be said about this policy since there is no measure of the risk associated with its adoption. Application of the Chebyshev inequality gives zero as the lower bound of the probability that the total demand for airlift support will be met. Likewise, zero is the lower bound of the probability that the demand for logistic support will be met. Such a policy should not have much appeal to decision makers since they are faced with an undetermined amount of risk.

Using the estimates, the deterministic equivalent for the chance-constrained, input-output model is given by

$$\min \quad 35x_1 + 25x_2$$

$$\begin{aligned} \text{subject to} \quad .75x_1 - .3x_2 - 1000 - w_1v_1 &\geq 0 \\ -.15x_1 + .8x_2 - 1500 - w_2v_2 &\geq 0 \\ v_1^2 - .0025x_1^2 - .0009x_2^2 + .0009x_1x_2 - 10000 &\geq 0 \\ v_2^2 - .0001x_1^2 - .0004x_2^2 + .0001x_1x_2 - 12000 &\geq 0 \\ u_1(w_1^2 + 1) - 1 &\geq 0 \\ u_2(w_2^2 + 1) - 1 &\geq 0 \\ .05 - u_1 &\geq 0 \\ .1 - u_2 &\geq 0 \end{aligned}$$

$$.1 - u_1 - u_2 \geq 0$$
$$6500 - x_1 - .5x_2 \geq 0$$
$$x_1, x_2, u_1, u_2, v_1, v_2, w_1, w_2 \geq 0$$

Using Fiacco and McCormick's nonlinear programming algorithm [17] to solve this problem*, one gets $x_1 = 3909.8$ and $x_2 = 3310.4$ as the solution which gives $219,603,000 as the expected cost. What else can be said about establishing this level for the decision variables? One can say that the probability that the total demand for airlift support will be met is at least .95, and the probability that the total demand for logistic support will be met is at least .9 (in fact, it is at least .95), and the probability that total demand is met is at least .9. Clearly, much of the uncertainty associated with the previous policy has been eliminated.

Since some of the advantages of using the chance-constrained model have been stated, the major disadvantage must also be pointed out. One desirable feature of the Chebyshev inequality is that it holds regardless of the distribution of the random variables being measured. But this feature can clearly work against anyone using it in the above model, since it can induce a decision maker to select a more cautious policy than one he might normally select. This action would most certainly increase his costs. To see this, one should remember that the Chebyshev inequality is used to obtain a lower bound on the probability that some

*See Appendix B.

particular constraint be met. If this lower bound is set equal to some appropriate number, say .95 for example, it is quite possible that the true measure of the probability is higher than this bound. In fact, the true probability could be extremely close to one; it is in the example, if one assumes that each constraint has an underlying normal distribution.

This undesirable feature must be considered the price one must pay for not knowing the distributions of the random variables associated with the model. As with any model, it does not answer all questions that a decision maker might have concerning the uncertainty he faces; however, it certainly gives him insight into his problems. Also, the model may be reformulated to be of greater use to the decision maker. Such a formulation could be

$$\min \quad u_1 + u_2$$

$$\text{subject to} \quad P[(1 - a_{11})x_1 - a_{12}x_2 \geq D_1] \geq 1 - u_1$$

$$P[-a_{21}x_1 + (1 - a_{22})x_2 \geq D_2] \geq 1 - u_2$$

$$P\left[\begin{bmatrix}(1 - a_{11}) & -a_{12} \\ -a_{21} & (1 - a_{22})\end{bmatrix}\begin{bmatrix}x_1 \\ x_2\end{bmatrix} \geq \begin{bmatrix}D_1 \\ D_2\end{bmatrix}\right] \geq 1 - u_1 - u_2$$

$$P[c_1x_1 + c_2x_2 \leq c_B] \geq 1 - \beta$$

$$x_1 + .5x_2 \leq 6500$$

$$x_1,\ x_2,\ u_1,\ u_2 \geq 0,$$

where the random variables are as previously defined, and c_B is the budgeted cost ceiling placed on the model. This model has the interpretation that one is attempting to minimize the risk of not meeting total demand, subject to the constraint that the probability that total costs do not exceed the budgeted ceiling, is greater than 1 - β. Clearly, this model would be of great value to the decision maker since he is getting the greatest amount of risk aversion for each dollar he spends, plus he is given an upper bound on the measure of the risk that he must face by adopting this policy.

It has been demonstrated that the chance-constrained programming model can be applied to the input-output analysis model. Clearly, a chance-constrained, input-output analysis model could be developed for the U. S. Air Force and could be used for budgeting and force structure problems. However, a competent analyst might ask whether the application of chance-constrained programming to input-output analysis is contrived. Or he might ask whether the insights gained by using the model are worth the computational burdens that the model places on him. To answer these questions, he must consider the advantages of using the model. One major advantage is that is forces the decision maker to recognize the risk and uncertainty that he faces, and it gives him a direct control over some of this risk. Furthermore, the model is "distribution free" since the analyst need not know the probability distribution of the random variables involved; he only need know estimations of the mean and variance of each random variable. The model can readily be used for budgeting or personnel allocation problems. Finally, the model is particularly applicable for optimal economic planning in organizations which are comprised of interrelated groups, where these groups supply goods and services to each other and to activities outside the organization. In this light, chance-constrained, input-output analysis can be considered as mathemetical programming from the systems management point of view.

Bibliography

1. Agnew, N. H., R. A. Agnew, J. Rassmussen, and K. R. Smith. "An Application of Chance Constrained Programming to Portfolio Selection in a Casualty Insurance Firm." Management Science, 15: B-512--B-520 (1969).

2. Agnew, R. A. "Inequalities With Application in Economic Risk Analysis." To appear in Journal of Applied Probability.

3. -----. "Mean-Variance Certainty Equivalents for Expected Logarithmic Utility." Submitted for publication.

4. -----. "Simple Bounds With Application in Stochastic Critical Path Analysis." Paper presented at Operations Research Society of America meeting, April 1970, Washington, D. C.

5. Baumol, W. J. Economic Theory and Operations Analysis. (Second Edition) Englewood Cliffs, N. J.: Prentice Hall, Inc., 1965.

6. Charnes, A. and W. W. Cooper. "Chance-Constrained Programming." Management Science, 6: 73-89 (1959).

7. ----- and -----. "Deterministic Equivalents for Optimizing and Satisficing Under Chance-Constraints." Operations Research, 11: 18-39 (1963).

8. ----- and -----. "Chance Constraints and Normal Deviates." Journal American Statistical Association, 57: 134-148 (1962).

9. -----, ----- and G. H. Symonds. "Cost Horizons and Certainty Equivalents: An Approach to Stochastic Programming of Heating Oil Production." Management Science, 4: 235-263 (1958).

10. -----, ----- and G. L. Thompson. "Constrained Generalized Medians and Hypermedians as Deterministic Equivalents for Two-Stage Linear Programs Under Uncertainty." Management Science, 12: 83-112 (1965).

Bibliography

11. -----, ----- and -----. "Critical Path Analysis via Chance-Constrained and Stochastic Programming." Operations Research, 12: 460-470 (1964).

12. ----- and M. J. L. Kirby. "Some Special P-Models in Chance-Constrained Programming." Management Science, 14: 183-195 (1967).

13. Dantzig, G. Linear Programming and Extensions. Princeton: Princeton University Press, 1963.

14. Dorfman, R., P. A. Samuelson, and R. M. Solow. Linear Programming and Economic Analysis. New York: McGraw-Hill, 1958.

15. Eisner, M. J., R. S. Kaplan, and J. V. Soden. "Admissible Decision Rules for the E-Model of Chance-Constrained Programming." Management Science, 17: 337-353 (1971).

16. Feller, W. An Introduction to Probability Theory and Its Applications. Vol. II (Second Edition). New York: John Wiley and Sons, Inc., 1971.

17. Fiacco, A. and G. McCormick. Nonlinear Programming: Sequential Unconstrained Minimization Techniques. New York: John Wiley and Sons, Inc., 1968.

18. Hillier, F. S. "Chance-Constrained Programming With 0-1 or Bounded Continuous Decision Variables." Management Science, 14: 34-57 (1967).

19. ------. "The Evaluation of Risky Interrelated Investments." Technical Report No. 73, Contract Nonr-225 (53), July 24, 1964.

20. ----- and G. J. Lieberman. Introduction to Operations Research. San Francisco: Holden-Day, 1967.

21. Karlin, S. and W. J. Studden. Tchebycheff Systems: With Applications in Analysis and Statistics. New York: Interscience Publishers, 1966.

22. Kataoka, S. "A Stochastic Programming Model." Econometrica, 31: 181-196 (1963).

Bibliography

23. Leontief, W. and Others. Studies in the Structure of the American Economy. New York: Oxford University Press, 1953.

24. Miller, B. L. and H. M. Wagner. "Chance-Constrained Programming With Joint Constraints." Operations Research, 13: 930-945 (1965).

25. Pyle, S. H. and S. J. Turnovsky. "Risk Aversion in Chance Constrained Portfolio Selection." Management Science, 18: 218-225 (1971).

26. ----- and -----. "Safety-First and Expected Utility Maximinization in Mean-Standard Deviation Portfolio Analysis." Review of Economics and Statistics, 52: 75-81 (1970).

27. Raike, W. M. "Dissection Methods for Solutions in Chance Constrained Programming Problems Under Discrete Distributions." Management Science, 16: 708-715 (1970).

28. Roy, A. D. "Safety-First and the Holding of Assets." Econometrica, 20: 431-449 (1952).

29. Sengupta, J. K. "Safety-First Rules Under Chance-Constrained Linear Programming." Operations Research, 17: 112-132 (1969).

30. Seppala, Y. "Constructing Sets of Uniformly Tighter Linear Approximations for a Chance Constraint." Management Science, 17: 736-749 (1971).

31. Tintner, G. "Stochastic Linear Programming With Applications to Agricultural Economics." Second Symposium on Linear Programming. Washington, D. C.: National Bureau of Standards, 1963.

Appendix A

Mathematical Development of Deterministic Constraints

Suppose one is given the chance-constraint

$$P[y_i = \Sigma_{j=1}^{n} a_{ij}x_j - b_i \leq 0] \geq 1 - \alpha_i,$$

where y_i has a known probability distribution with mean $\mu_i = \mu_i(x_1, \ldots, x_n)$ and variance $\sigma_i^2 = \sigma_i^2(x_1, \ldots, x_n)$. The problem is to develop a deterministic equivalent constraint for the above probabilistic constraint.

$$P[\Sigma_{j=1}^{n} a_{ij}x_j \leq b_i] = P[y_i \leq 0]$$

$$= P[(y_i - \mu_i)/\sigma_i \leq -\mu_i/\sigma_i] \geq 1 - \alpha_i.$$

This probabilistic constraint holds if and only if

$$-\mu_i/\sigma_i \geq F^{-1}(1 - \alpha_i) = \bar{K}_{1 - \alpha_i},$$

where $F(.)$ is the cumulative distribution function of the random variable $(y_i - \mu_i)/\sigma_i$. This deterministic constraint can be rendered more tractible by adding a variable to give

$$\mu_i + K_{1 - \alpha_i} v_i \leq 0$$

$$v_i^2 - \sigma_i^2 \geq 0$$

$$v_i \geq 0.$$

Use of the Chebyshev Inequality to Give Deterministic Equivalent Constraints for Both Single and Joint Chance-Constraints

Suppose one is given the single and joint chance-constraints

$$P[y_i \geq 0] \geq 1 - \alpha_i \quad i = 1, 2, \ldots, m$$

$$P \cap_{i=1}^{m} (y_i \geq 0) \geq 1 - \alpha,$$

where $E(y_i) = \mu_i$ and Var $(y_i) = \sigma_i^2$. Consider

$$P[y_i \geq 0] = 1 - P[y_i < 0] = 1 - P[\mu_i - y_i > \mu_i].$$

But the one-sided Chebyshev inequality gives the following bound

$$P[\mu_i - y_i > \mu_i] \leq \sigma_i^2/(\sigma_i^2 + \mu_i^2), \text{ provided } \mu_i \geq 0.$$

This implies that

$$P[y_i \geq 0] \geq 1 - \sigma_i^2/(\sigma_i^2 + \mu_i^2).$$

The constraint $P[y_i \geq 0] \geq 1 - \alpha_i$ holds if and only if $\sigma_i^2/(\sigma_i^2 + \mu_i^2) \leq \alpha_i$. Now consider the joint constraint

$$P \cap_{i=1}^{m} (y_i \geq 0) = 1 - P \cup_{i=1}^{m} (y_i < 0)$$

$$= 1 - P \cup_{i=1}^{m} [\mu_i - y_i > \mu_i] \geq 1 - \Sigma_{i=1}^{m} P[\mu_i - y_i > \mu_i]$$

which follows from the subadditivity property of any probability measure. Again using the Chebyshev inequality one has

$$P \cap_{i=1}^{m} (y_i \geq 0) \geq 1 - \Sigma_{i=1}^{m} \sigma_i^2/(\sigma_i^2 + \mu_i^2) \geq 1 - \alpha .$$

The constraint holds if and only if

$$\Sigma_{i=1}^{m} \sigma_i^2/(\sigma_i^2 + \mu_i^2) \leq \alpha.$$

Utilizing these inequalities, the deterministic equivalent constraints can be written as

$$\begin{array}{l} t_i \leq \alpha_i \\ \mu_i \geq 0 \end{array} \quad i = 1, 2, \ldots, m.$$

$$\Sigma_{i=1}^{m} t_i \leq \alpha,$$

where $t_i = \sigma_i^2/(\sigma_i^2 + \mu_i^2)$.

Appendix B

Nonlinear Programming

This section outlines the nonlinear programming algorithm used by the author to solve the deterministic equivalent, chance-constrained, input-output analysis model in chapter IV. It is Fiacco and McCormick's interior point, unconstrained algorithm. To understand its use, consider a nonlinear programming model which can be formulated as

$$\text{Min} \quad h(x_1, \ldots, x_n)^*$$

$$\text{subject to} \quad g_i(x_1, \ldots, x_n) \geq 0 \quad i = 1, 2, \ldots, m.$$

In their interior point algorithm, Fiacco and McCormick transform the constrained problem into an unconstrained minimization problem by using some transformation of the form

$$W(x_1, \ldots, x_n, r) = h(x_1, \ldots, x_n) + s(r) \Sigma_{i=1}^{m} I(g_i(x_1, \ldots, x_n)),$$

where $s(r)$ is some function of r which has the property $s(r) \to 0$ as $r \to 0$, and $I(g_i(x_1, \ldots, x_n))$ is some function of the i-th constraint which has the property $I(g_i(x_1, \ldots, x_n)) \to \infty$ as $g_i(x_1, \ldots, x_n) \to 0$. In solving the problem in chapter IV, the functions, s and I, were taken as

$$s(r) = r, \text{ and } I(g_i(x_1, \ldots, x_n)) = -\ln g_i(x_1, \ldots, x_n).$$

*If one's objective is max $f(x_1, \ldots, x_n)$, one must define $h(x_1, \ldots, x_n) = -f(x_1, \ldots, x_n)$ in order to use Fiacco and McCormick's algorithm.

This gives the problem

$$\min \; W(x_1, \ldots, x_n, r) = h(x_1, \ldots, x_n) - r \; \Sigma_{i=1}^{m} \ln \left[g_i(x_1, \ldots, x_n)\right].$$

Basically, the method of solution is to solve this problem for arbitrary positive r, decrease r (but keeping it positive) and solve the problem again. Continuing in this fashion, one will generate a sequence of values $[x_i^{(k)}]$ for $i = 1, 2, \ldots, n$. If $r^{(k)} \to 0$ as $k \to \infty$, then $x_i^{(k)} \to x_i^*$ as $k \to \infty$, $i = 1, 2, \ldots, n$, where x_i^* is the i-th element of the vector x^* which is a point that will give a local minimum to the constrained problem.

Fiacco and McCormick prove that if the functions $f, g_1, \ldots, g_m$ are continuous and if one follows the above procedure in minimizing $W(x_1, \ldots, x_n, r)$, the sequence of solutions will converge to a locally optimal vector, x^*, for the constrained problem

$$\min \; h(x_1, \ldots, x_n)$$
$$\text{subject to} \quad g_i(x_1, \ldots, x_n) \geq 0 \quad i = 1, 2, \ldots, m$$

provided a local minimum exists. If $h(x_1, \ldots, x_n)$ is also a convex function and $g_i(x_1, \ldots, x_n)$ is a concave function, $i = 1, 2, \ldots, m$, the convergence is global. Unfortunately, this is not the case for the problem in chapter IV.

The Algorithm

Expressing the constrained problem as the following unconstrained function

$$W(x_1, \ldots, x_n, r) = h(x_1, \ldots, x_n)$$
$$- r \Sigma_{i=1}^{m} \ln g_i(x_1, \ldots, x_n)$$

the solution to the constrained problem is obtained by using the algorithm outlined below.

Step 1. An initial vector $x^{(0)} = (x_1^{(0)}, \ldots, x_n^{(0)})$ is chosen such that $g_i(x^{(0)}) > 0$ for $i = 1, 2, \ldots, m$.

Step 2. An initial value for r_1 is chosen, in this case $r_1 = 1$ is used.

Step 3. Using $x^{(k-1)}$ and r_k, the function $W(x_1, \ldots, x_n, r_k)$ is minimized over n space giving $x^{(k)}$. The steepest descent algorithm is utilized for this purpose, although the Newton method could also be used. However, with the Newton method, one must compute the Hessian matrix of $W(x_1, \ldots, x_n, r)$, and this becomes impractical as the number of variables and the number of constraints increases.

Step 4. Terminate computations if the new solution is acceptable; if not, continue.

Step 5. Select a new value for r_{k+1}. $r_{k+1} = r_k/2$ is used in the problem.

Step 6. Using the new values for $(x_1, \ldots, x_n)$ and r, return to step 3.

Minimizing a Function Using the Steepest Descent Method

The steepest descent algorithm which can be used to minimize an unconstrained function is an iterative technique which utilizes only the first partial derivatives. Suppose one desires to minimize the function $W(x, r_k)$ with respect to the vector x while holding r_k fixed. Using the steepest descent method, one selects an arbitrary vector $x^{(0)}$ and computes $\nabla W(x^{(0)}, r_k)$, the gradient vector of W with respect to the vector x; then a new value for the vector x is found by using the iterative relation

$$x^{(i+1)} = x^{(i)} - \lambda_i \nabla W(x^{(i)}, r_k),$$

where λ_i is the smallest non-negative value which locally minimizes W along $-\nabla W(x^{(i)}, r_k)$ starting from $x^{(i)}$. λ_i is usually found by finding an upper and lower bound on it and then shrinking the interval of these bounds until they converge to the desired point.

The algorithm is terminated when the norm of the gradient vector is suitably small.

Vita

Jon R. Thomas was born in Milford, Delaware on 19 June 1943. He graduated from North High School in Akron, Ohio in 1961. He attended Kent State University and received a bachelor of science degree in mathematics from there in June, 1965. He entered the USAF in May, 1966 and received a commission in August, 1966. He graduated from the Communications Offices course in May, 1967 and served in that capacity with Headquarters, 10 AF for three years. From there, he entered the Air Force Institute of Technology in June, 1970.

Permanent Address: 1172 Mt. Vernon Avenue
Akron, Ohio

This thesis was typed by Mrs. Marcia Muhlenkamp.